Several Symmetric Inequalities of Exponential Kind

Arkady Alt

In this article we suggest a general approach for proving certain symmetric inequalities of exponential kind in three variables which have appeared in print at various times.

Theorem 1 Let n, m, p, and q be arbitrary nonnegative real numbers, such that $n \geq m$ and $p \geq q$. Then for any positive real numbers a, b, c the following inequality holds

$$\frac{a^{n+p}+b^{n+p}+c^{n+p}}{a^{m+q}+b^{m+q}+c^{m+q}} \geq \frac{a^n+b^n+c^n}{a^m+b^m+c^m}\cdot\frac{a^p+b^p+c^p}{a^q+b^q+c^q}\,.$$

Proof: Let $\sigma(x) = \sigma(x;a,b,c) = \sum\limits_{\text{cyclic}} a^x$; the inequality then becomes

$$\frac{\sigma\,(n+p)}{\sigma\,(m+q)} \geq \frac{\sigma\,(n)}{\sigma\,(m)}\cdot\frac{\sigma\,(p)}{\sigma\,(q)}\,.$$

The inequality is essentially the same upon switching n and p or m and q, so we may suppose that $n \geq p$ and $m \geq q$. Then $q = \min\{n,\,m,\,p,\,q\}$.

Since the inequality to be proved is equivalent to $\sigma\,(n+p)\,\sigma\,(m)\,\sigma\,(q) \geq \sigma\,(m+q)\,\sigma\,(n)\,\sigma\,(p)$ and we also have

$$\begin{aligned}
&\sigma\,(n+p)\,\sigma\,(m)\,\sigma\,(q)\\
&\quad= \sum_{\text{cyclic}} a^{n+p}\cdot\left(\sum_{\text{cyclic}} a^{m+q} + \sum_{\text{cyclic}}\left(a^m b^q + b^m a^q\right)\right)\\
&\quad= \left(\sum_{\text{cyclic}} a^{n+p}\right)\left(\sum_{\text{cyclic}} a^{m+q}\right) + \sum_{\text{cyclic}}\left(a^{n+p}+b^{n+p}\right)\left(a^m b^q + b^m a^q\right)\\
&\qquad+ \sum_{\text{cyclic}} c^{n+p}\left(a^m b^q + b^m a^q\right),
\end{aligned}$$

with the analogous inequality holding for $\sigma\,(m+q)\,\sigma\,(n)\,\sigma\,(p)$, it therefore suffices to prove the following two inequalities:

$$\sum_{\text{cyclic}}\left(a^{n+p}+b^{n+p}\right)\left(a^m b^q + b^m a^q\right) \geq \sum_{\text{cyclic}}\left(a^{m+q}+b^{m+q}\right)\left(a^n b^p + b^n a^p\right),$$

$$\sum_{\text{cyclic}} c^{n+p}\left(a^m b^q + b^m a^q\right) \geq \sum_{\text{cyclic}} c^{m+q}\left(a^n b^p + b^n a^p\right).$$

The first inequality above is settled by the following calculation:

$$\begin{aligned}
&\sum_{\text{cyclic}} \left(a^{n+p}+b^{n+p}\right)\left(a^m b^q + b^m a^q\right) \\
&- \sum_{\text{cyclic}} \left(a^{m+q}+b^{m+q}\right)\left(a^n b^p + b^n a^p\right) \\
= \;& \sum_{\text{cyclic}} \big(a^{n+p+m}b^q + b^{n+p+m}a^q + a^m b^{n+p+q} + b^m a^{n+p+q} \\
&\qquad - a^{n+m+q}b^p - b^{n+m+q}a^p - a^n b^{m+p+q} - b^n a^{m+p+q}\big) \\
= \;& \sum_{\text{cyclic}} a^q b^q \left(a^{n+m+p-q} + b^{n+m+p-q} - a^{n+m}b^{p-q} - b^{n+m}a^{p-q}\right) \\
&+ \sum_{\text{cyclic}} a^m b^m \left(a^{n+p+q-m} + b^{n+p+q-m} - a^{p+q}b^{n-m} - b^{p+q}a^{n-m}\right) \\
= \;& \sum_{\text{cyclic}} a^q b^q \left(a^{n+m} - b^{n+m}\right)\left(a^{p-q} - b^{p-q}\right) \\
&+ \sum_{\text{cyclic}} a^m b^m \left(a^{p+q} - b^{p+q}\right)\left(a^{n-m} - b^{n-m}\right) \;\geq\; 0\,.
\end{aligned}$$

Lastly, since

$$\begin{aligned}
\sum_{\text{cyclic}} c^{n+p}\left(a^m b^q + b^m a^q\right) &= \sum_{\text{cyclic}} c^q\left(a^{n+p}b^m + b^{n+p}a^m\right); \\
\sum_{\text{cyclic}} c^{m+q}\left(a^n b^p + b^n a^p\right) &= \sum_{\text{cyclic}} c^q\left(a^{m+p}b^n + b^{m+p}a^n\right),
\end{aligned}$$

the second inequality that remains to be proved now follows immediately from

$$\begin{aligned}
&\sum_{\text{cyclic}} c^q\left(a^{n+p}b^m + b^{n+p}a^m - a^{m+p}b^n - b^{m+p}a^n\right) \\
= \;& \sum_{\text{cyclic}} a^m b^m c^q \left(a^{n-m+p} + b^{n-m+p} - a^p b^{n-m} - b^p a^{n-m}\right) \\
= \;& \sum_{\text{cyclic}} a^m b^m c^q \left(a^p - b^p\right)\left(a^{n-m} - b^{n-m}\right) \;\geq\; 0\,.
\end{aligned}$$

■

Corollary 1 Let k be a nonnegative integer and let $p \geq q \geq 0$. Then for any positive real numbers a, b, and c the following inequality holds

$$\frac{a^{kp}+b^{kp}+c^{kp}}{a^{kq}+b^{kq}+c^{kq}} \;\geq\; \left(\frac{a^p+b^p+c^p}{a^q+b^q+c^q}\right)^k .$$

Proof: We set $n = kp$, $m = kq$ in Theorem 1 to obtain

$$\frac{\sigma(kp+p)}{\sigma(kq+q)} \;\geq\; \frac{\sigma(kp)}{\sigma(kq)} \cdot \frac{\sigma(p)}{\sigma(q)}$$

and that yields the inequality

$$\frac{\sigma((k+1)p)}{\sigma((k+1)q)}\left(\frac{\sigma(p)}{\sigma(q)}\right)^{-(k+1)} \;\geq\; \frac{\sigma(kp)}{\sigma(kq)}\left(\frac{\sigma(p)}{\sigma(q)}\right)^{-k},$$

which implies that

$$\frac{\sigma(kp)}{\sigma(kq)}\left(\frac{\sigma(p)}{\sigma(q)}\right)^{-k} \;\geq\; \frac{\sigma(1\cdot p)}{\sigma(1\cdot q)}\left(\frac{\sigma(p)}{\sigma(q)}\right)^{-1} \;=\; 1,$$

and the inequality to be proved now follows. ∎

Theorem 2 Let a, b, and c be positive real numbers. Then for any positive integer n the function

$$L_n(x) \;=\; L_n(x;a,b,c) \;=\; \frac{a^n+b^n+c^n}{a^{nx}+b^{nx}+c^{nx}}\sum_{\text{cyclic}}\left(\frac{a^x}{b+c}\right)^n$$

is increasing in x on $(0,\infty)$.

Proof: Let p, $q \in (0,\infty)$ and $q < p$. Due to the homogeneity of $L_n(x;a,b,c)$ with respect to a, b, and c, it suffices to prove the assertion when $a+b+c=1$. Using the expansion $\frac{1}{(1-t)^n} = \sum_{k=0}^{\infty}\binom{k+n-1}{n-1}t^k$ we obtain

$$\begin{aligned}
&\frac{\sigma(np)\sigma(nq)}{\sigma(n)}\big(L_n(p)-L_n(q)\big)\\
&\quad=\; \sigma(nq)\sum_{\text{cyclic}}\frac{a^{np}}{(1-a)^n} \;-\; \sigma(np)\sum_{\text{cyclic}}\frac{a^{nq}}{(1-a)^n}\\
&\quad=\; \sigma(nq)\sum_{\text{cyclic}}\sum_{k=0}^{\infty}\binom{k+n-1}{n-1}a^{k+np} - \sigma(np)\sum_{\text{cyclic}}\sum_{k=0}^{\infty}\binom{k+n-1}{n-1}a^{k+nq}\\
&\quad=\; \sum_{k=0}^{\infty}\binom{k+n-1}{n-1}\big(\sigma(nq)\sigma(k+np)-\sigma(np)\sigma(k+nq)\big)\\
&\quad=\; \sum_{k=0}^{\infty}\binom{k+n-1}{n-1}\sum_{\text{cyclic}}\big(a^{k+np}b^{nq}+a^{nq}b^{k+np}-a^{k+nq}b^{np}-a^{np}b^{k+nq}\big)\\
&\quad=\; \sum_{k=0}^{\infty}\binom{k+n-1}{n-1}\sum_{\text{cyclic}}a^{nq}b^{nq}\left(a^{n(p-q)}-b^{n(p-q)}\right)\left(a^k-b^k\right) \;\geq\; 0,
\end{aligned}$$

since $\left(a^{n(p-q)}-b^{n(p-q)}\right)\left(a^k-b^k\right) \geq 0$ for any nonnegative integer k. ∎

Corollary 2 For any positive real numbers a, b, c, r and any positive numbers p and q such that $q < r < p$ the following inequality holds

$$\frac{1}{\sigma(nq)}\sum_{\text{cyclic}}\left(\frac{a^q}{b^r+c^r}\right)^n \leq \frac{1}{\sigma(nr)}\sum_{\text{cyclic}}\left(\frac{a^r}{b^r+c^r}\right)^n \leq \frac{1}{\sigma(np)}\sum_{\text{cyclic}}\left(\frac{a^p}{b^r+c^r}\right)^n.$$

Proof: Since $L_n(x; a^r, b^r, c^r)$ is increasing in x and $q < r < p$, we have

$$L_n\left(\frac{q}{r}; a^r, b^r, c^r\right) \ \le\ L_n\left(1; a^r, b^r, c^r\right) \ \le\ L_n\left(\frac{p}{r}; a^r, b^r, c^r\right),$$

which is equivalent to the inequality to be proved. ∎

By the results of Corollary 1 and Corollary 2 we obtain successively

$$\frac{1}{\sigma(nq)}\sum_{\text{cyclic}}\left(\frac{a^q}{b^r + c^r}\right)^n \ \le\ \frac{1}{\sigma(nr)}\sum_{\text{cyclic}}\left(\frac{a^r}{b^r + c^r}\right)^n ;$$

$$\frac{\displaystyle\sum_{\text{cyclic}}\left(\frac{a^q}{b^r + c^r}\right)^n}{\displaystyle\sum_{\text{cyclic}}\left(\frac{a^r}{b^r + c^r}\right)^n} \ \ge\ \frac{\sigma(nr)}{\sigma(nq)} \ \ge\ \left(\frac{\sigma(nr)}{\sigma(nq)}\right)^n ,$$

and similarly we obtain

$$\frac{\displaystyle\sum_{\text{cyclic}}\left(\frac{a^p}{b^r + c^r}\right)^n}{\displaystyle\sum_{\text{cyclic}}\left(\frac{a^r}{b^r + c^r}\right)^n} \ \ge\ \frac{\sigma(np)}{\sigma(nr)} \ \ge\ \left(\frac{\sigma(p)}{\sigma(r)}\right)^n .$$

It follows that for any positive real numbers a, b, c, r and any positive real numbers p, q such that $q < r < p$, the following inequality holds

$$\frac{1}{\sigma^n(q)}\sum_{\text{cyclic}}\left(\frac{a^q}{b^r + c^r}\right)^n \le \frac{1}{\sigma^n(r)}\sum_{\text{cyclic}}\left(\frac{a^r}{b^r + c^r}\right)^n \le \frac{1}{\sigma^n(p)}\sum_{\text{cyclic}}\left(\frac{a^p}{b^r + c^r}\right)^n .$$

Corollary 3 Let a, b, c be positive real numbers and let

$$F(x) \ =\ F(x; a, b, c) \ =\ \frac{a + b + c}{a^x + b^x + c^x}\sum_{\text{cyclic}}\frac{a^x + b^x}{a + b},$$

$$E(x) \ =\ E(x; a, b, c) \ =\ \frac{1}{a^x + b^x + c^x}\sum_{\text{cyclic}}\frac{a\left(b^x + c^x\right)}{b + c}.$$

Then $F(x)$ and $E(x)$ are each decreasing on $(0, \infty)$.

Proof: We have

$$L_1(x) \ =\ \frac{\sigma(1)}{\sigma(x)}\sum_{\text{cyclic}}\frac{\sigma(x)}{b + c} - \frac{\sigma(1)}{\sigma(x)}\sum_{\text{cyclic}}\frac{b^x + c^x}{b + c} \ =\ \sum_{\text{cyclic}}\frac{a + b + c}{b + c} - F(x),$$

hence, $F(x)$ is decreasing on $(0, \infty)$ because $L_1(x)$ is increasing on $(0, \infty)$ by Theorem 2. Straightforward calculations show that $E(x) = F(x) - 2$, hence $E(x)$ is also decreasing on $(0, \infty)$. ∎

We now apply the preceding results to obtain some generalizations of various problems.

Problem For any positive real numbers a, b, c, r and any positive real numbers p, q such that $q < r < p$ prove the following inequalities:

$$\frac{1}{\sigma(p)}\sum_{\text{cyclic}}\frac{a^p+b^p}{a^r+b^r} \;\le\; \frac{3}{\sigma(r)} \;\le\; \frac{1}{\sigma(q)}\sum_{\text{cyclic}}\frac{a^q+b^q}{a^r+b^r}\,; \tag{1}$$

$$\frac{1}{\sigma(p)}\sum_{\text{cyclic}}\frac{a^r(b^p+c^p)}{b^r+c^r} \;\le\; 1 \;\le\; \frac{1}{\sigma(q)}\sum_{\text{cyclic}}\frac{a^r(a^q+b^q)}{a^r+b^r}\,. \tag{2}$$

Solution: We have $F(\frac{p}{r};a^r,b^r,c^r) \le F(1;a^r,b^r,c^r) \le F(\frac{q}{r};a^r,b^r,c^r)$ by Corollary 2, and since $F(1;a^r,b^r,c^r) = 3$ the first inequality follows.

Similarly, $E(\frac{p}{r};a^r,b^r,c^r) \le E(1;a^r,b^r,c^r) \le E(\frac{q}{r};a^r,b^r,c^r)$ and since $E(1;a^r,b^r,c^r) = 1$ the second inequality follows. ■

Inequality (1) is a generalization of the inequality $\sum_{\text{cyclic}}\frac{a^2+b^2}{a+b} \le \frac{3\sigma(2)}{\sigma(1)}$ in [2], and also a generalization of the inequality in [3].

Inequality (2) generalizes the inequality $\sum_{\text{cyclic}}\frac{x^p(y+z)}{y^p+z^p} \ge x+y+z$, for positive x, y, z, and $p > 1$, which is Peter Woo's generalization of the inequality in [4] (see the commentary on p. 180). Furthermore, by using the rightmost relation of Inequality (2) we can obtain a generalization of the inequality $\sum_{\text{cyclic}}\frac{a^{\lambda+1}}{b^\lambda+c^\lambda} \ge \frac{a+b+c}{2}$, for $\lambda \ge 0$, suggested by Walther Janous in [4] (again, see the commentary on p. 180). Namely: for any positive real numbers a, b, c, p, and q the following inequality holds

$$\sum_{\text{cyclic}}\frac{a^{p+q}}{b^p+c^p} \;\ge\; \frac{a^q+b^q+c^q}{2}\,. \tag{3}$$

Proof: The inequality $\frac{a^{p+q}(b^q+c^q)}{b^{p+q}+c^{p+q}} \le \frac{2a^{p+q}}{b^p+c^p}$ holds since simple manipulations show that it is equivalent to $(b^q-c^q)(b^p-c^p) \ge 0$, and from inequality (2) it follows that $\sum_{\text{cyclic}}\frac{a^{p+q}(b^q+c^q)}{b^{p+q}+c^{p+q}} \ge a^q+b^q+c^q$, hence,

$$\sum_{\text{cyclic}}\frac{a^{p+q}}{b^p+c^p} \;\ge\; \frac{1}{2}\sum_{\text{cyclic}}\frac{a^{p+q}(b^q+c^q)}{b^{p+q}+c^{p+q}} \;\ge\; \frac{a^q+b^q+c^q}{2}\,,$$

which proves inequality (3).

In [1] the inequality $\sum_{\text{cyclic}}\left(\frac{c^2}{a^2+b^2}\right)^n \ge \sum_{\text{cyclic}}\left(\frac{c}{a+b}\right)^n$ was suggested. The next theorem offers a generalization.

Theorem 3 Let n be a positive integer and a, b, c be positive real numbers. Then $G(x) = G_n(x; a, b, c) = \sum_{\text{cyclic}} \left(\frac{c^x}{a^x + b^x}\right)^n$ is increasing on $(0, \infty)$.

Proof: Let $p > q > 0$ and let $A_x = \frac{a^x}{\sigma(x)}$, $B_x = \frac{b^x}{\sigma(x)}$, and $C_x = \frac{c^x}{\sigma(x)}$. Then we obtain

$$G_n(p) \geq G_n(q) \iff \sum_{\text{cyclic}} \frac{A_p^n}{(1 - A_p)^n} \geq \sum_{\text{cyclic}} \frac{A_q^n}{(1 - A_q)^n}$$

$$\iff \sum_{\text{cyclic}} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} A_p^{k+n} \geq \sum_{\text{cyclic}} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} A_q^{k+n}$$

$$\iff \sum_{k=1}^{\infty} \binom{k+n-1}{n-1} \sum_{\text{cyclic}} A_p^{k+n} \geq \sum_{k=1}^{\infty} \binom{k+n-1}{n-1} \sum_{\text{cyclic}} A_q^{k+n}$$

$$\iff \sum_{k=1}^{\infty} \binom{k+n-1}{n-1} \frac{\sigma\big((k+n)p\big)}{\sigma^{k+n}(p)} \geq \sum_{k=1}^{\infty} \binom{k+n-1}{n-1} \frac{\sigma\big((k+n)q\big)}{\sigma^{k+n}(q)},$$

and the last inequality above holds termwise by the result of Corollary 1. ■

By applying the result of Theorem 3 to the terms of an infinite series we obtain the following corollary.

Corollary 4 Let $h(t) = \sum_{n=0}^{\infty} h_n t^n$, where each h_n is nonnegative and the series converges for $t \geq 0$. Then for any positive real numbers a, b, c the function $G_h(x; a, b, c) = \sum_{\text{cyclic}} h\left(\frac{c^x}{a^x + b^x}\right)$ is increasing in x on $(0, \infty)$.

References

[1] Razvan Satnianu, Problem 11080, *American Mathematical Monthly*, Vol. 111, No. 4.

[2] Nguyen Le Dung, Problem 221.5, "All the best from Vietnamese Problem Solving Journals", *The Mathscope*, Feb. 12 (2007) p. 5.

[3] Arkady Alt, Problem 3300, ***CRUX Mathematicorum with Mathematical Mayhem***, Vol. 33, No. 8 (2007) p. 489.

[4] Sefket Arslanagic, Problem 2927*, ***CRUX Mathematicorum with Mathematical Mayhem***, Vol. 30, No. 3 (2004) p. 172; solution in Vol. 31, No. 3 (2005) pp. 179–180.

Arkady Alt
San Jose , California
USA
arkady.alt@gmail.com